

Abelian and center gauge fixing in continuum Yang-Mills-Theory for general gauge groups¹

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Abstract

A prescription for center gauge fixing for pure Yang-Mills theory in the continuum with general gauge groups is presented. The emergence of various types of singularities (magnetic monopoles and center vortices) appearing in the course of the gauge fixing procedure are discussed.

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1 Introduction

There are two promising mechanisms to explain color confinement: the dual Meissner effect [1, 2, 3] and the picture of condensation of center vortices [4, 5]. The relevant infrared degrees of freedom corresponding to these two mechanisms are magnetic monopoles and center vortices, respectively. Both can be identified in partial gauge fixings where they arise as defects to the gauge fixing, magnetic monopoles and center vortices respectively arise in Abelian gauges and center gauges. Both pictures of confinement have received support from recent lattice calculations performed in specific gauges to identify the relevant infrared degrees of freedom. Monopole dominance in the string tension [7, 8, 9] has been found in maximally Abelian gauge and in all forms of the Abelian gauges considered monopole condensation occurs in the confinement phase and is absent in the de-confinement phase [10]. Lattice calculations performed in the so-called maximum center gauge show that the vortex content detected after center projection produces virtually the full string tension, while the string tension disappears, if the center vortices are removed from the lattice ensemble [11, 12]. This property of center dominance exists at finite temperature [13]. The vortices have also been shown to condense in the confinement phase [14]. Furthermore in the gauge field ensemble devoid of center vortices chiral symmetry breaking disappears and all field configurations belong to the topologically trivial sector [12]. The continuum version of maximum center gauge has been derived in [15].

Both gauge fixing procedures, the maximally Abelian gauge as well as the maximum center gauge, suffer from the Gribov problem [16]. To circumvent the Gribov problem, the Laplacian gauge [17], the Laplacian Abelian gauge [18] and the Laplacian center gauge [19, 20] have been introduced. In the Laplacian Abelian and Laplacian center gauge one uses (usually the two lowest-lying) eigenfunctions ψ_1 and ψ_2 of the covariant Laplace operator transforming in the adjoint representation of the gauge group to define the gauge. The Laplacian center gauge can be understood as the extension of the Laplacian Abelian gauge. For gauge group $SU(2)$ the Abelian gauge is fixed by demanding that at every point x in space-time the gauge fixed field $\psi_1^V(x)$ points into the positive 3-direction in color space. To fix the residual Abelian gauge freedom up to the center \mathbb{Z}_2 of the gauge group one rotates the second to lowest eigenvector $\psi_2^V(x)$ into the 1–3-plane in color space. Actually it is not important that ψ_1 and ψ_2 are eigenfunctions of the covariant Laplace operator - the only

important point is that ψ_1 and ψ_2 homogeneously transform under gauge transformations. Merons and instantons configurations have been studied in this gauge in [21]. Recently de Forcrand and Pepe [22] extended the Laplacian center gauge to $SU(N)$ gauge groups. The aim of the present paper is to further extend the Laplacian center gauge to arbitrary gauge groups. As an example we consider the symplectic group $Sp(2)$ which is the universal covering group of $SO(5)$. The latter group is relevant in connection with string theories and superconductivity [27]

2 Lie algebra conventions

We denote by \mathbf{G} and \mathcal{G} the gauge group and its Lie algebra, respectively, and by $\mathbf{H} \subset \mathbf{G}$ and $\mathcal{H} \subset \mathcal{G}$ its Cartan subgroup and subalgebra, respectively. The group \mathbf{G} is assumed to be simple and has rank r . We use the following Lie algebra conventions. We denote by H_k , $k = 1, \dots, r$ an orthogonal basis in the Cartan subalgebra $\mathcal{H} \subset \mathcal{G}$ normalized to

$$\text{Tr}(H_k H_l) = \delta_{kl}, \quad (2.1)$$

and satisfying

$$[H_k, E_\alpha] = \alpha_k E_\alpha, \quad (2.2)$$

where E_α is the root vector⁴ to the root α . We denote by Greek letters ρ, σ, \dots vectors in the Cartan subalgebra

$$\rho = \sum_{k=1}^r \rho_k H_k \in \mathcal{H}, \quad \rho_k \in \mathbb{R}, \quad k = 1, \dots, r \quad (2.3)$$

for which we define the scalar product by

$$(\rho, \sigma) = \text{Tr}(\rho \sigma) = \sum_{k=1}^r \rho_k \sigma_k \quad (2.4)$$

where the last relation follows from (2.1). Furthermore, let

$$\alpha_{(i)}, \quad \mu_{(i)}, \quad \alpha_{(i)}^\vee = \frac{2\alpha_{(i)}}{(\alpha_{(i)}, \alpha_{(i)})}, \quad \mu_{(i)}^\vee = \frac{2\mu_{(i)}}{(\alpha_{(i)}, \alpha_{(i)})} \quad (2.5)$$

⁴We work in the complexified Lie algebra. The real Lie Algebra is spanned by the elements iH_k , $i(E_\alpha + E_{-\alpha})$ and $(E_\alpha - E_{-\alpha})$.

be the simple roots, fundamental weights, co-roots and co-weights, respectively, satisfying

$$(\alpha_{(i)}^\vee, \mu_{(j)}) = (\alpha_{(i)}, \mu_{(j)}^\vee) = \delta_{ij}. \quad (2.6)$$

We denote by Σ , Σ^+ and Π the sets of all roots, all positive roots and all simple roots, respectively. For later use we also note that the co-weight vectors $\mu_{(i)}^\vee$ generate the center elements via

$$z_i = \exp(2\pi i \mu_{(i)}^\vee), i = 1, \dots, r \quad (2.7)$$

which form the center of the group \mathbf{G} , whereas for the co-roots $\alpha_{(i)}^\vee$ we have

$$\mathbb{1} = \exp(2\pi i \alpha_{(i)}^\vee), i = 1, \dots, r. \quad (2.8)$$

Note that the lattice generated by the co-roots $\alpha_{(i)}^\vee, i = 1, \dots, r$ is a subset of the lattice generated by the co-weights $\mu_{(i)}^\vee, i = 1, \dots, r$.

3 Abelian gauge fixing

Before presenting the Laplacian center gauge fixing for arbitrary Lie groups it is worth while to introduce its Abelian counter part, the (Laplacian) Abelian gauge fixing for general Lie groups, which is part of the (Laplacian) center gauge fixing. To this end we consider a Lie algebra valued field⁵ ψ_1 in the adjoint representation transforming homogeneously under gauge transformations. In the following we will refer to such a field as ‘‘Higgs field’’. We fix the gauge by requiring that for every $x \in M$ with M being the space-time manifold $\psi_1^V(x)$ is in some closed convex subset \mathcal{F} (to be specified below) of the Cartan subalgebra \mathcal{H} , i.e. we are looking for a gauge transformation V with

$$\psi_1^V(x) = V(x)^{-1} \psi_1(x) V(x) = h(x) \quad \text{and} \quad h(x) \in \mathcal{F} \subset \mathcal{H} \quad \forall x \in M. \quad (3.1)$$

Let us emphasize that it is not sufficient to require $\psi_1^V(x)$ to be an element of the Cartan subalgebra \mathcal{H} . This would leave the group of Weyl reflections \mathbf{W} unfixed, which is given by reflections in \mathcal{H} at planes through the origin perpendicular to a root. In fact if $w \in \mathbf{W}$ then $V(x)w$ also rotates $\psi_1(x)$ into the Cartan subalgebra but the transformed field $\psi_1^{(Vw)}(x)$ will in general differ from $\psi_1^V(x)$ ($\psi_1^{(Vw)}(x)$ is the image of $\psi_1^V(x)$ under the Weyl reflection

⁵ ψ_1 is a field in the complexified Lie algebra such that $\exp(i\psi_1) \in \mathbf{G}$, e.g. for $\mathbf{G} = SU(N)$ the field ψ_1 takes values in the set of hermitian matrices.

$w \in \mathbf{W}$). Therefore, in order to fix the gauge transformed image of $\psi_1(x)$ uniquely one has to restrict $\psi_1(x) = h(x)$ to the so-called fundamental domain $\mathcal{F} \subset \mathcal{H}$ which is given by the coset \mathcal{H}/\mathbf{W} , i.e. the fundamental domain \mathcal{F} is obtained by identifying all vectors of the Cartan subalgebra \mathcal{H} which are related by Weyl reflections $w \in \mathbf{W}$. It is well known that the Cartan subalgebra decomposes into Weyl chambers related to each other by Weyl reflections and the fundamental domain \mathcal{F} can be identified with a specific Weyl chamber, which we choose as

$$\mathcal{F} = \{ \rho \mid (\rho, \alpha_{(i)}) \geq 0 \text{ for all simple roots } \alpha_{(i)} \} . \quad (3.2)$$

From equation (2.6) we obtain that every $\rho \in \mathcal{F}$ can be uniquely written as a linear combination of the fundamental co-weights $\mu_{(i)}^\vee$ with real and positive coefficients

$$\rho \in \mathcal{F} \Leftrightarrow \rho = \sum_{k=1}^r c_k \mu_{(k)}^\vee \quad \text{with } c_k = (\rho, \alpha_{(k)}) \geq 0, \quad k = 1, \dots, r. \quad (3.3)$$

From its definition (3.1) it follows that the matrix $V(x)$ is defined only up to right-multiplication with a matrix $g(x)$ commuting with $h(x)$

$$V(x) \rightarrow V(x)g(x), \quad g(x)h(x)g(x)^{-1} = h(x). \quad (3.4)$$

The set of all such matrices $g(x)$ form a subgroup of \mathbf{G} , the centralizer of $h(x)$ in \mathbf{G} , denoted by $\mathbf{C}_{h(x)}(\mathbf{G})$. The centralizer contains the Cartan subgroup \mathbf{H} of \mathbf{G} . At points x where the centralizer is just \mathbf{H} we can choose $V(x)$ and $h(x)$ smoothly, assuming the Higgs field $\psi_1(x)$ was smooth before gauge fixing. However, at points $x \in M$ for which $\psi_1^V(x) = h(x)$ is on the boundary of the fundamental domain \mathcal{F} the residual gauge freedom is enlarged, i.e. the centralizer $\mathbf{C}_{h(x)}(\mathbf{G})$ becomes non-Abelian and as a consequence there are obstructions to a smooth choice of $V(x)$ and $h(x)$. The set of such singular points (or gauge fixing defects)

$$\mathcal{D}_{\text{ag}} := \{ x \in M \mid \mathbf{C}_{h(x)} \neq \mathbf{H} \} \quad (3.5)$$

is referred to as Abelian gauge fixing defect manifold. Generically the defect manifold consists of connected subsets of co-dimension 3, i.e. they form points in $D = 3$ and lines in $D = 4$ and represent magnetic monopoles and monopole loops, respectively.

To illustrate that the centralizer of a point ρ on the boundary of \mathcal{F} is non-Abelian, we decompose ρ as in equation (3.3). For ρ on the boundary of \mathcal{F} at least one of the

coefficients c_k vanishes, say $c_l = 0$. Then it follows

$$[\rho, E_{\pm\alpha_{(l)}}] = \sum_k \rho_k [H_k, E_{\pm\alpha_{(l)}}] = \sum_k \rho_k (\pm(\alpha_{(l)})_k) E_{\pm\alpha_{(l)}} \quad (3.6)$$

$$= \pm(\rho, \alpha_{(l)}) E_{\pm\alpha_{(l)}} = \pm c_l E_{\pm\alpha_{(l)}} = 0 \quad (3.7)$$

implying that the $SU(2)$ -subgroup of \mathbf{G} generated by $E_{\pm\alpha_{(l)}}$ and $\alpha_{(l)}$ is contained in the centralizer of ρ , i.e. the centralizer is non-Abelian. If there are more than one vanishing coefficients c_k in (3.3), the centralizer becomes larger. A complete classification of the various types of possible centralizers can be found in ref. [23].

To identify magnetic monopoles and their charges we introduce the magnetic gauge potential A_{mag} and its field strength F_{mag}

$$A_{\text{mag}} := V^{-1} dV|_{\mathcal{H}}, \quad F_{\text{mag}} = dA_{\text{mag}} = -V^{-1} dV \wedge V^{-1} dV|_{\mathcal{H}}, \quad (3.8)$$

where $|_{\mathcal{H}}$ denotes projection onto the Cartan subalgebra \mathcal{H} . The gauge potential A_{mag} transforms as a gauge potential with respect to the residual Abelian gauge transformations, see equation (3.4). The magnetic charge of a defect is given by the flux through a closed surface \mathcal{S} surrounding the defect⁶ \mathcal{S} and integrate F_{mag} over \mathcal{S}

$$Q_{\text{mag}} = \frac{1}{2\pi i} \int_{\mathcal{S}} F_{\text{mag}} \quad (3.9)$$

The magnetic charge is quantized [23]

$$Q_{\text{mag}} = \sum_k n_k \alpha_{(k)}^{\vee}, \quad n_k \in \mathbb{Z}, \quad k = 1, \dots, r, \quad (3.10)$$

which is a generalization of the relation found in refs. [24, 25, 26]. Let us consider a defect at x_0 such that only one of the coefficients c_k , say c_l , in the decomposition

$$h(x_0) = \sum_{k=1}^r c_k \mu_{(k)}^{\vee}, \quad c_k > 0 \quad \text{for } k \neq l, \quad c_l = 0 \quad (3.11)$$

vanishes. Then the charge Q_{mag} of the defect is an integer multiple of $\alpha_{(l)}^{\vee}$. In general one can show that the coefficient n_k in (3.10) is zero if the coefficient c_k in (3.11) is non-zero.

⁶In $D = 4$ the defect is generically a line. In this case one takes a 3-dimensional space \mathcal{K} traversing the monopole line in exactly one point, say x_0 , and chooses \mathcal{S} to be a closed surface in \mathcal{K} surrounding x_0 .

4 Center gauge fixing

Above we have performed the Abelian gauge fixing in which the Higgs field ψ_1 has been gauge transformed into the fundamental domain $\mathcal{F} \subset \mathcal{H}$

$$\psi_1^V(x) = h(x) \in \mathcal{F}. \quad (4.1)$$

After this first step we are left with a residual Abelian $\mathbf{H} \cong U(1)^r$ gauge freedom (away from defect points). This residual gauge freedom will now be fixed up to the center of the gauge group. This is done by rotating the non-Cartan part of the Higgs field $\psi_2^V(x)$ in the way described below:

We decompose the Higgs field $\psi_2^V(x) \in \mathcal{G}$ with respect to a basis of the Lie algebra \mathcal{G} :

$$\begin{aligned} \psi_2^V(x) &= \sum_{k=1}^r h_k(x) \alpha_{(k)} \\ &+ \sum_{k=1}^r \left(e_{\alpha_{(k)}}(x) E_{\alpha_{(k)}} + e_{\alpha_{(k)}}(x)^* E_{-\alpha_{(k)}} \right) \\ &+ \sum_{\beta \in \Sigma^+, |\beta| > 1} (f_\beta(x) E_\beta + f_\beta(x)^* E_{-\beta}), \end{aligned} \quad (4.2)$$

where $e_{\alpha_{(k)}}(x)$ and $f_\beta(x)$ are complex numbers and the star denotes complex conjugation. The first term on the r.h.s. represents the part of $\psi_2^V(x)$ lying in the Cartan subalgebra and the summation runs here over all simple roots. Furthermore the second term represents the contributions from the simple root vectors. The center gauge fixing condition which fixes the residual Abelian (Cartan) gauge symmetries up to the center of the gauge group is chosen by requiring all coefficients of the simple roots, $e_{\alpha_{(k)}}(x)$, $k = 1, \dots, r$ to be real and positive.

We will show that this requirement can be obtained by an Abelian gauge transformation, leaving unfixed only the center of the gauge group. Indeed an element $g(x) \in \mathbf{H}$ of the Cartan subgroup can be parameterized by

$$g(x) = \exp \left(i \sum_k s_k(x) \mu_{(k)}^\vee \right) \quad (4.3)$$

where $\mu_{(k)}^\vee$ are the co-weights and the $s_k(x)$ are real numbers.

Adjoint action of $g(x)$ on $E_{\pm\alpha_{(k)}}$ yields by using equations (2.2) and (2.6), i.e.

$$[\mu_{(l)}^\vee, E_{\alpha_{(k)}}] = (\mu_{(l)}^\vee)_p [H_p, E_{\alpha_{(k)}}] = (\mu_{(l)}^\vee)_p (\alpha_{(k)})_p E_{\alpha_{(k)}} \quad (4.4)$$

$$= (\mu_{(l)}^\vee, \alpha_{(k)}) E_{\alpha_{(k)}} = \delta_{kl} E_{\alpha_{(k)}} , \quad (4.5)$$

the following relation

$$g(x)^{-1} E_{\pm\alpha_{(k)}} g(x) = \exp(\mp i s_k(x)) E_{\pm\alpha_{(k)}} . \quad (4.6)$$

This means that the coefficients $e_{\alpha_{(k)}}(x)$ in equation (4.2) transform under gauge transformation with $g(x)$ as

$$e_{\alpha_{(k)}}(x) \rightarrow \exp(-i s_k(x)) e_{\alpha_{(k)}}(x) . \quad (4.7)$$

If $e_{\alpha_{(k)}}(x) \neq 0$ the condition that $e_{\alpha_{(k)}}(x)$ to be real and positive after gauge rotation with $g(x)$ determines the parameters $s_k(x)$ modulo 2π . But $\exp(2\pi i \mu_{(k)}^\vee)$ is a center element (c.f. equation (2.7)) of \mathbf{G} . Therefore, requiring the coefficients of the simple roots, $e_{\alpha_{(k)}}(x)$, to be real and positive fixes the group element $g(x)$ up to an element of the center of \mathbf{G} . This shows that the above proposed gauge condition indeed fixes the gauge group \mathbf{G} up to its center.

4.1 The center gauge fixing for specific gauge groups

Below we illustrate the above proposed center gauge fixing for two specific gauge groups.

4.1.1 The gauge group $SU(3)$

For gauge group $SU(3)$ this type of center gauge fixing is the one suggested by de Forcrand and Pepe [22]. In this case the Higgs field $\psi_2^V(x)$ has the matrix representation

$$\psi_2^V = \begin{pmatrix} h_1 & e_{\alpha_{(1)}} & f_{\alpha_{(1)}+\alpha_{(2)}} \\ e_{\alpha_{(1)}}^* & (-h_1 + h_2) & e_{\alpha_{(2)}} \\ f_{\alpha_{(1)}+\alpha_{(2)}}^* & e_{\alpha_{(2)}}^* & -h_2 \end{pmatrix} \quad (4.8)$$

and the coefficients $e_{\alpha_{(k)}}(x)$ are simply the elements on the secondary diagonal of the matrix $\psi_2^V(x) \in su(3)$. After center gauge fixing these coefficients are demanded to be real and

positive. Gauge rotation with

$$g = \exp(i(s_1\mu_{(1)}^\vee + s_2\mu_{(2)}^\vee)) \quad (4.9)$$

$$= \text{diag}(\exp(i/3(2s_1 + s_2)), \exp(i/3(-s_1 + s_2)), \exp(i/3(-s_1 - 2s_2))) \quad (4.10)$$

transforms ψ_2^V into

$$g^{-1}\psi_2^V g = \begin{pmatrix} h_1 & e_{\alpha_{(1)}}e^{-is_1} & f_{\alpha_{(1)}+\alpha_{(2)}}e^{-i(s_1+s_2)} \\ e_{\alpha_{(1)}}^*e^{is_1} & (-h_1 + h_2) & e_{\alpha_{(2)}}e^{-is_2} \\ f_{\alpha_{(1)}+\alpha_{(2)}}^*e^{i(s_1+s_2)} & e_{\alpha_{(2)}}^*e^{is_2} & -h_2 \end{pmatrix}. \quad (4.11)$$

Obviously, if $e_{\alpha_{(1)}} \neq 0$ and $e_{\alpha_{(2)}} \neq 0$ then s_1 and s_2 are defined modulo 2π . Since

$$\exp(2\pi i\mu_{(1)}^\vee) = \text{diag}(e^{4/3\pi i}, e^{-1/3\pi i}, e^{-1/3\pi i}) = e^{4/3\pi i} \mathbb{1} \quad (4.12)$$

$$\exp(2\pi i\mu_{(2)}^\vee) = \text{diag}(e^{1/3\pi i}, e^{-2/3\pi i}, e^{-2/3\pi i}) = e^{1/3\pi i} \mathbb{1} \quad (4.13)$$

are just the center elements of $SU(3)$ this implies that g is defined up to the center of $SU(3)$.

4.1.2 The gauge group $Sp(2)$

Below we will discuss the gauge group $Sp(2)$ which is the universal covering group of $SO(5)$ which is relevant in the context string theory, see e.g. [27]. The symplectic group $Sp(2)$ is defined as the group of linear transformations in two-dimensional quaternionic space leaving invariant the sesquilinear form

$$(\mathbf{a}, \mathbf{b}) = \bar{\mathbf{a}}^t \mathbf{b} = \bar{a}_1 b_1 + \bar{a}_2 b_2 \in \mathbb{H}, \quad (4.14)$$

where \mathbf{a} is a quaternionic vector with the two quaternionic components a_1 and a_2 . Here the bar denotes quaternionic conjugation and t denotes transposition. A group element g of $Sp(2)$ is a 2×2 matrix with quaternionic entries such that $\bar{g}^t g = \mathbb{1}$. A Lie algebra element A of $sp(2)$ fulfills $\bar{A}^t + A = 0$. If we represent quaternions by 2×2 matrices:

$$q = q_0 \mathbb{1} + iq_k \sigma_k \in \mathbb{H}, \quad \bar{q} = q_0 \mathbb{1} - iq_k \sigma_k \quad (4.15)$$

the elements of $Sp(2)$ are 4×4 matrices made of 2×2 quaternionic blocks. Then the decomposition (4.2) of the Higgs field ψ_2^V in terms of the roots reads

$$\psi_2^V = \begin{pmatrix} (h_1 - \frac{1}{2}h_2) & e_{\alpha(1)} & ie_{\alpha(2)}^* \frac{1}{\sqrt{2}} & if_{\alpha(1)+\alpha(2)} \frac{1}{\sqrt{2}} \\ e_{\alpha(1)}^* & (-h_1 + \frac{1}{2}h_2) & -if_{\alpha(1)+\alpha(2)}^* \frac{1}{\sqrt{2}} & ie_{\alpha(2)} \frac{1}{\sqrt{2}} \\ -ie_{\alpha(2)} \frac{1}{\sqrt{2}} & if_{\alpha(1)+\alpha(2)} \frac{1}{\sqrt{2}} & \frac{1}{2}h_2 & f_{\alpha(1)+2\alpha(2)} \\ -if_{\alpha(1)+\alpha(2)}^* \frac{1}{\sqrt{2}} & -ie_{\alpha(2)}^* \frac{1}{\sqrt{2}} & f_{\alpha(1)+2\alpha(2)}^* & -\frac{1}{2}h_2 \end{pmatrix} \quad (4.16)$$

A short calculation shows that gauge rotation with

$$\begin{aligned} g &= \exp(i(s_1\mu_{(1)}^\vee + s_2\mu_{(2)}^\vee)) \\ &= \text{diag}(\exp(i/2 s_1), \exp(-i/2 s_1), \exp(i/2 (s_1 + 2s_2)), \exp(-i/2 (s_1 + 2s_2))) \end{aligned} \quad (4.17)$$

yields

$$e_{\alpha(1)} \rightarrow e_{\alpha(1)} e^{-is_1} \quad (4.18)$$

$$e_{\alpha(2)} \rightarrow e_{\alpha(2)} e^{-is_2} \quad (4.19)$$

as expected from equation (4.7). Again s_1 and s_2 are fixed modulo 2π . Since furthermore

$$\exp(2\pi i\mu_{(1)}^\vee) = -\mathbb{1}, \quad (4.20)$$

$$\exp(2\pi i\mu_{(2)}^\vee) = \mathbb{1} \quad (4.21)$$

(where $\mathbb{1}$ denotes the 4-dimensional unit matrix) represent the center elements of $Sp(2)$ it follows that the center of $Sp(2)$ is unfixed by our gauge condition.

One further remark is here in order. One could also try to fix the gauge by demanding $e_{\alpha(1)}$ and $f_{\alpha(1)+2\alpha(2)}$ to be real and positive. This corresponds to considering the $SU(2) \times SU(2)$ subgroup of $Sp(2)$ (given by diagonal quaternionic matrices) and center gauge fixing in each of the two $SU(2)$ subgroups. In this case the residual gauge freedom would be the product of the centers of the two $SU(2)$ subgroups. But this would imply that the residual gauge freedom would be bigger than the center of $Sp(2)$ - the gauge rotation g would be

fixed up to multiplication with one of the four group elements

$$\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \quad \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \quad \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad (4.22)$$

(where $\mathbb{1}$ denotes now the 2×2 unit matrix) while the center of the group $Sp(2)$ is given by the first two matrices only.

5 Discussion of the gauge fixing defects

In the course of the above described gauge fixing procedure different types of singularities can appear.

Magnetic monopole defects arise whenever the residual gauge freedom after the first step of the gauge fixing procedure is larger than $\mathbf{H} \cong U(1)^r$, i.e. if the centralizer of $\psi_1^V(x)$ is larger than $U(1)^r$. This is equivalent to $\psi_1^V(x)$ lying on the boundary of the fundamental domain \mathcal{F} .

Vortex defects arise whenever the residual gauge freedom, which is left after the second step of the gauge fixing procedure, is larger than the center of \mathbf{G} . This is the case whenever one of the coefficients $e_{\alpha_{(k)}}(x)$ in the decomposition (4.2) is zero⁷. The set of those points for which this happens is referred to in the following as center gauge fixing defect manifold

$$\mathcal{D}_{\text{cg}} := \left\{ x \in M \mid e_{\alpha_{(k)}}(x) = 0 \text{ for at least one } k, 1 \leq k \leq r \right\} \quad (5.1)$$

since at these points the above described gauge fixing procedure is ill-defined. It is easy to see that connected subsets of this defect manifold form vortices. Indeed, the equation $e_{\alpha_{(k)}}(x) = 0$ implies two conditions (one for the real and one for the imaginary part of $e_{\alpha_{(k)}}(x)$). Therefore vortex defects have co-dimension 2, i.e. vortex defects are 1-dimensional lines in $D = 3$ and 2-dimensional faces in $D = 4$. If $e_{\alpha_{(k)}}(x) = 0$ then the coefficient $s_k(x)$ in the decomposition (4.3) is not fixed and the residual gauge freedom is larger than the center of \mathbf{G} (the group of residual gauge transformations contains e.g. the $U(1)$ subgroup generated by $\mu_{(k)}^\vee$).

Now we will show that the magnetic monopoles lie on top of vortices by construction. Let $\psi_1^V(x_0) = \rho$ be on the boundary of \mathcal{F} such that at least one of the coefficients in

⁷For gauge group $SU(2)$ this means that at such points $\psi_1(x)$ and $\psi_2(x)$ are linearly dependent.

the decomposition (3.3), say $c_l(x_0)$, vanishes. Then the centralizer $\mathbf{C}_\rho(\mathbf{G})$ contains the $SU(2)$ -subgroup \mathbf{K} generated by $E_{\pm\alpha_{(l)}}$ and $\alpha_{(l)}$, c.f. equation (3.7). Indeed we can choose a matrix $g \in \mathbf{K} \subset \mathbf{C}_\rho(\mathbf{G})$ such that in the decomposition (4.2) of $\psi_2^{Vg}(x_0)$ the coefficient $e_{\alpha_{(l)}}(x_0)$ is zero, which means that x_0 is also a defect of the center gauge fixing. In this sense we obtain $\mathcal{D}_{\text{ag}} \subset \mathcal{D}_{\text{cg}}$. On the other hand from a physical point of view it is clear that from a magnetic monopole with non-zero charge there must emanate a vortex (or Dirac string) carrying away the magnetic flux - and the vortex singularities introduced by our gauge fixing procedure indeed carry magnetic flux as we will show below. More rigorously one can argue as follows:

Let us assume there is a magnetic monopole with non-zero charge at x_0 and let us consider a surface \mathcal{S} surrounding x_0 such that the integral of F_{mag} over \mathcal{S} is non-zero. Then it is impossible to choose the gauge transformation Vg smoothly on the whole surface \mathcal{S} . On the other hand let us assume that there is no vortex singularity traversing \mathcal{S} . But then, because \mathcal{S} is simply connected, it is possible to choose the gauge transformation Vg smoothly on \mathcal{S} . But this contradicts the assumption that the magnetic monopole at x_0 has non-zero charge. Hence, there must be vortex singularities traversing \mathcal{S} . As we can choose \mathcal{S} arbitrarily closed to x_0 we can conclude that a vortex singularity emanates from the magnetic monopole at x_0 .

In the following we will show that the flux of A_{mag} (3.8) carried by a vortex is quantized - it is given by a linear combination of the co-weights with integer valued coefficients. For this purpose we will integrate A_{mag} along an infinitesimal loop \mathcal{C} surrounding the vortex singularity⁸ away from magnetic monopoles. We obtain

$$\Phi = \frac{1}{2\pi i} \int_{\mathcal{C}} A_{\text{mag}} = \frac{1}{2\pi i} \int_{\mathcal{C}} ((Vg)^{-1}d(Vg))|_{\mathcal{H}} \quad (5.2)$$

$$= \frac{1}{2\pi i} \int_{\mathcal{C}} (g^{-1}VdVg + g^{-1}dg)|_{\mathcal{H}}. \quad (5.3)$$

The integral over $V^{-1}dV$ (first term in the second line) vanishes, because we can choose V smoothly on the whole path \mathcal{C} , if the path is away from magnetic monopoles. Hence we

⁸In $D = 4$ the vortex is a two-dimensional surface. In this case one takes a 2-dimensional face \mathcal{K} intersecting the vortex (singularity) sheet at exactly one point x_0 and chooses \mathcal{C} as a loop in \mathcal{K} surrounding x_0 .

obtain

$$\Phi = \frac{1}{2\pi i} \int_{\mathcal{C}} (g^{-1} dg)_{|\mathcal{H}} = \frac{1}{2\pi} \int_{\mathcal{C}} \left(\sum_{k=1}^r ds_k \mu_{(k)}^{\vee} \right) \quad (5.4)$$

$$= \frac{1}{2\pi} \sum_{k=1}^r \Delta s_k \mu_{(k)}^{\vee}, \quad (5.5)$$

where Δs_k is an integer multiple of 2π . This is because g at the starting point of the loop \mathcal{C} can differ from g at the endpoint only by a center element (2.7) of \mathbf{G} (s_k is fixed modulo 2π by the gauge fixing conditions).

Integration of F_{mag} over an infinitesimal surface \mathcal{S} surrounding a magnetic monopole yields the continuity equation for magnetic flux. The magnetic charge of the monopole is the sum of the fluxes of all vortex singularities emanating from the monopole:

$$Q_{\text{mag}} = \sum_l \Phi_l. \quad (5.6)$$

The center gauge fixing can be visualized geometrically in a bundle picture. Appending to each $x \in \mathcal{D}_{\text{cg}}^c = M \setminus \mathcal{D}_{\text{cg}}$ all matrices $V(x)g(x)$, which transform the Higgs fields ψ_1 and ψ_2 in the demanded way, we get a principal bundle P_{cg} with structure group $\mathbf{Z}(\mathbf{G})$, the center of \mathbf{G} , because $V(x)g(x)$ is fixed up to multiplication with center elements. If the center of \mathbf{G} has v elements, then P_{cg} is a v -fold covering of $\mathcal{D}_{\text{cg}}^c$. In analogy to complex function theory we may look at P_{cg} as a Riemann surface of a multivalued function. The vortex singularities can be identified as branching points. Now we can classify the different vortex singularities. We consider a closed loop \mathcal{C} surrounding the vortex singularity and lift this loop into the covering manifold P_{cg} . There are v different classes of lifted loops - the starting point and the endpoint of the lifted loops have to differ by one of the v center elements, say z . If the center element z is the identity (then the lifted loop is closed) we call the singularity a Dirac string - otherwise we call it a center vortex. The center element z is simply given by

$$z = \exp(2\pi i \Phi), \quad (5.7)$$

where Φ is the magnetic flux through the vortex. On the other hand the center element z

is obviously equal to the Wilson loop⁹

$$z = \mathcal{P} \exp \left(\int_{\mathcal{C}} \omega (Vg)^{-1} d(Vg) \right) \omega = (Vg)^{-1} d(Vg). \quad (5.8)$$

We are interested in a globally well defined gauge transformation Vg on $\mathcal{D}_{\text{cg}}^c$. But if we go once around a center vortex along a path \mathcal{C} then the gauge transformation Vg has to jump by a center element at one point on the path \mathcal{C} . This means that we have to introduce cuts connecting the center vortices and on these cuts the gauge transformation Vg jumps by a center element.

We can get rid of these cut singularities by working in the adjoint representation, i.e. by working with the gauge group $\mathbf{G}/\mathbf{Z}(\mathbf{G})$ instead of \mathbf{G} . In this case the set $\{g, gz_1, gz_2, \dots\}$ of matrices in \mathbf{G} differing by multiplication with center elements z_1, z_2, \dots are identified as the element \hat{g} in the group $\mathbf{G}/\mathbf{Z}(\mathbf{G})$. But this means that the cuts where Vg jumps by a center element in the fundamental representation are invisible in the adjoint representation $\hat{V}\hat{g}$. As a consequence the gauge transformation $\hat{V}\hat{g}$ can be chosen smoothly everywhere except at center vortices, Dirac strings and magnetic monopoles. But the type of a center vortex singularity is no longer encoded in the Wilson loop¹⁰ (5.8) along the path \mathcal{C} surrounding the vortex, since this Wilson loop always equals the identity in $\mathbf{G}/\mathbf{Z}(\mathbf{G})$. In the adjoint representation the type of a center vortex can be related to an element of the first homotopy group $\pi_1(\mathbf{G}/\mathbf{Z}(\mathbf{G}))$ of the group $\mathbf{G}/\mathbf{Z}(\mathbf{G})$. To make this explicit we define a closed path $\hat{\mathcal{C}}$ in the Lie group $\mathbf{G}/\mathbf{Z}(\mathbf{G})$ defined by

$$\hat{\mathcal{C}}(t) := \mathcal{P} \exp \left(\int_0^t (\hat{V}(\mathcal{C}(\tau))\hat{g}(\mathcal{C}(\tau)))^{-1} \frac{d(\hat{V}(\mathcal{C}(\tau))\hat{g}(\mathcal{C}(\tau)))}{d\tau} d\tau \right), \quad (5.9)$$

where \mathcal{C} is a loop in the space time M surrounding the center vortex and it is parameterized by $t, 0 \leq t \leq 2\pi$. Obviously we obtain $\hat{\mathcal{C}}(2\pi) = \hat{\mathcal{C}}(0) = \mathbb{1}_{\text{adj}}$ which means that the path $\hat{\mathcal{C}}$ is closed in $\mathbf{G}/\mathbf{Z}(\mathbf{G})$. But the loop $\hat{\mathcal{C}}$ is not contractible in $\mathbf{G}/\mathbf{Z}(\mathbf{G})$, if \mathcal{C} surrounds a center vortex. In the adjoint representation the vortex is classified by the homotopy class of the loop $\hat{\mathcal{C}}$.

⁹The 1-form $\omega = (Vg)^{-1}d(Vg)$ is invariant under changing (Vg) by a center element z , i.e. under the transformation $(Vg) \rightarrow (Vg)z, z \in \mathbf{Z}(\mathbf{G})$. Therefore the form ω can be defined smoothly on the whole path \mathcal{C} around the center vortex.

¹⁰Strictly speaking one has to replace ω in equation (5.8) by $\hat{\omega} = (\hat{V}\hat{g})^{-1}d(\hat{V}\hat{g})$, i.e. by ω in the adjoint representation.

6 Concluding remarks

In the present paper we have given a prescription for center gauge fixing for arbitrary gauge groups. The different types of singularities (magnetic monopoles and center vortices) appearing during the gauge fixing procedure have been discussed. Lattice calculations show that center vortices are the relevant infrared degrees of freedom to explain color confinement. In the continuum Yang-Mills theory the occurrence of center vortices depends on topological properties of the gauge group - center vortices are related (and classified) by elements of the center of the gauge group or by elements of its first homotopy group. From this point of view it would be interesting to analyze a gauge group which has a trivial center *and* a trivial homotopy group such as the exceptional groups G_2 or E_8 . Such groups should not show color confinement, if center vortices are the only relevant infrared degrees of freedom for confinement.

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